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A Novel approach to design Fractional Order Digital Differentiator using Power Function and Least-Square method

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Abstract: In this paper, we have proposed a technique for designing of fractional order digital differentiator by use of power function and least-square method. The input signal is altered into a power function by using Taylor series expansion and its fractional derivative is calculated using the Grunwald-Letnikov explanation. Then, fractional order digital differentiator is demonstrated as a finite impulse response (FIR) system, which produces fractional order derivative of the Grunwald-Letnikov category for a power function. The FIR system coefficients are achieved by using least square method. Two illustrations are used to demonstrate that the fractional derivative of the digital signals is calculated by using the suggested technique. The results of the second example divulges that the suggested method gives superior performance in comparison to existing filtering method.

Keywords: Digital differentiator, Fractional order differentiator (FOD), Finite impulse response (FIR), Fractional derivative, Least Square (LS) method, Grunwald-Letnikov definition.

I. INTRODUCTION

This paper describes a novel approach for designing fractional order digital differentiator. The fractional order calculus is a 300-years-old topic; the theory of fractional-order derivative was developed usually in the nineteenth century. Recent books provide a good source of references on fractional calculus [1], [2]. However, applying fractional-order calculus to dynamic systems control is just a recent focus of interest. The fractional order of differentiation and integration is useful in control system applications. Fractional Calculus is generalization of ordinary differentiation and integration to non-integer order i.e. taking real number powers of differentiation operator.

$$D^{v}f(x)=\frac{d^{v}f(x)}{dx^{v}} \qquad (1)$$

If v is a natural number then the case is called higher integer order differentiation. Positive real number corresponds to fractional order differentiation. The historical developments culminated in two calculi which are based on the work of Riemann and Liouville (RL) at the one side and on the work of Grunwald and Letnikov (GL) on the other. The classical form of fractional calculus is given by the Riemann-Liouville integral. It is given as follows [1]:

$$_{a}D_{t}^{-\alpha}f(t)={}_{a}I_{t}^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\tau)^{\alpha-1}f(\tau)\,d\tau \qquad (2)$$

The main interest of this paper is to design a fractional order digital differentiator by the use of power function and least-square technique. In our methodology the input signal is altered into a power function by using Taylor series expansion and its fractional derivative is calculated using the Grunwald-Letnikov definition. Then, a fractional order digital differentiator is demonstrated as a finite impulse response (FIR) system which yields fractional order derivative of the Grunwald-Letnikov type for a power function. Least-square technique is used to attain FIR system approximation to fractional order differentiator. Power function has many applications in polynomial based filter designing. This paper is organized as follows: In Section 2, computation of fractional derivative based on Talyor series expansion is defined. In Section 3, the power function based design of least square fractional order FIR differentiator is defined. In Section 4, two examples are presented to validate the effectiveness of the proposed design approach. Finally, conclusions are made.

II. COMPUTATION OF FRACTIONAL DERIVATIVE

From the last few times, the concept of fractional calculus has been used in many applications of signal processing. The exclusive feature of fractional calculus is its capability to generalization of integral and differential operators to non-integer order. The generalized continuous integral-differential operator in is as follows

$$_{a}D_{t}^{\alpha}=\begin{cases}\frac{d^{\alpha}}{dt^{\alpha}}, & \alpha>0\\ 1, & \alpha=0\\ \int_{a}^{t}(d\tau)^{\alpha}, & \alpha<0\end{cases} \qquad (3)$$

where $_{a}D_{t}^{\alpha}$ denotes integral-differential operator to compute the α^{th} order fractional differentiation and integration of the input signal with respect to t and α is the primary condition of the operation. Some of the standard definitions for this integral-differential operation are Riemann-Liouville, Grunewald- Letnikov and the Caputo definitions etc. In this paper, the

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Grunwald-Letnikov definition for the fractional order calculation is used which is as follows

$${}_aD_t^{\alpha} s(t) = \lim_{\Delta \to 0} \sum_{k=0}^{\alpha} \frac{(-1)^k C_k^{\alpha}}{\Delta^{\alpha}} s(t - k\Delta) \quad (4)$$

where C_k^{α} is the binomial coefficient. The value of C_k^{α} is given by using the relation in between Eulers, Gamma function and factorial, which is defined as

$$C_k^{\alpha} = \binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} = \begin{cases} 1 & k = 1 \\ \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{1.2.3..k} \end{cases} \quad (5)$$

where, $\Gamma(.)$ is the gamma function. The outcome of fractional derivative depends on the bound of the operator a. A common value for this bound is $a = 0$. Based on this operator the derivative of power function t^r is

$${}_0D_t^{\alpha} t^r = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} \quad (6)$$

If $s(t)$ is a given function in terms of power series expansion, its fractional order derivative can be calculated using equation (6). The fractional order derivative of a digital signal is calculated by relating the discrete time samples of the signal to continuous time signal $s(t)$. Any given function $s(t)$ can be represented in the polynomial of t using Taylor series expansion (Tseng (2001)), as

$$s(t) = \sum_{r=0}^{\infty} a_r t^r \quad (7)$$

where, $a_r = \frac{D^r s(0)}{r!}$ for t=0, the α^{th} order fractional derivative of s(t) is given as

$${}_0D_t^{\alpha} s(t) = \sum_{r=0}^{\infty} a_r D_t^{\alpha} t^r = \sum_{r=0}^{\alpha} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} \quad (8)$$

Now, assume that $t = nT$, where T is sampling period. $z(n) = s(nT)$ is the sampling of $s(t)$ at $t = n$. Let power function $s(n) = n^r, 0 \le n \le M-1$ and its fractional derivative $D_t^{\alpha} s(n)$ is given as

$$D^{\alpha} s(n) = \sum_{r=0}^{\infty} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} \quad (9)$$

If the signal $s(n)$ is delayed by a value I its fractional derivative $D_t^{\alpha} s(n-1)$ is given as

$$D^{\alpha} s(n-I) = \sum_{r=0}^{\infty} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} (n-I)^{r-\alpha} \quad (10)$$

The above equation shows the desired response of fractional order differentiator. In the following section, this result will be used to estimate the FIR filter output.

III. LEAST SQUARE DESIGN METHOD

In this section, we will use the results of $D^{\alpha} s(n)$ to compute the transfer function of fractional order differentiator, whose frequency response estimates the ideal frequency response of fractional order differentiator in (1). The transfer function of digital FIR filter can be written as

$$H(z) = \sum_{q=0}^{N} h(q) z^{-q} \quad (11)$$

We have to design a FIR filter $H(z)$ which is a digital differentiator with filter coefficients $h(q)$. When the signal $s(n)$ passes through Nth order FIR filter $H(z)$, its output $y(n)$ is given by

$$y(n) = \sum_{k=0}^{N} h(k) s(n-k) \quad (12)$$

since

$$s(n) = \sum_{r=0}^{\infty} a_r n^r$$
$$s(n-k) = \sum_{r=0}^{\infty} a_r (n-k)^r$$
$$y(n) = \sum_{r=0}^{\infty} a_r \sum_{k=0}^{N} h(k)\,(n-k)^r \quad (13)$$

To achieve the α^{th} order fractional derivative of $s(n)$, compute the filter coefficients $h(q)$ such that filter output $y(n)$ is identical to the delayed fractional derivative $D^{\alpha} s(n-I)$, that is

$$D^{\alpha} s(n-I) = h(n) * s(n)$$
$$D^{\alpha} s(n-I) = \sum_{r=0}^{\infty} a^r \sum_{k=0}^{N} h(k)\,(n-k)^r \quad (14)$$

From eq. (10) and (14), we obtain

$$\sum_{r=0}^{\infty} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} (n-I)^{r-\alpha} = \sum_{r=0}^{\infty} a^r \sum_{k=0}^{N} h(k)\,(n-k)^r \quad (15)$$

The comparison between the desired response and FIR filter output gives the error function $e(n)$, which can be written as

$$e(n) = \sum_{r=0}^{\infty} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} (n-I)^{r-\alpha} - \sum_{r=0}^{\infty} a^r \sum_{k=0}^{N} h(k)\,(n-k)^r$$

Using least square technique, the function to be minimized is

$$\varepsilon = \sum_{n=0}^{M-1} e^2(n)$$

$$= \sum_{n=0}^{M-1} . \left[\sum_{r=0}^{\infty} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} (n-I)^{r-\alpha} - \sum_{r=0}^{\infty} a^r \sum_{k=0}^{N} h(k)\,(n-k)^r \right]^2$$

$$= \sum_{n=0}^{M-1} . \left[\sum_{r=0}^{\infty} a_r C(r,\alpha)(n-I)^{r-\alpha} - \sum_{r=0}^{\infty} a^r \sum_{k=0}^{N} h(k)\,(n-k)^r \right]^2$$

where,

$$C(r,\alpha) = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}$$

$$\varepsilon = \sum_{r=0}^{N} \sum_{n=0}^{M-1} \left[a_r C(r,\alpha)(n-I)^{r-\alpha} - \sum_{k=0}^{N} a_r h(k)(n-k)^r \right]^2 \quad (16)$$

To reduce the least-square error, derivative of above equation must be zero, according to the optimization theory, which can be written as

$$\frac{\partial \varepsilon}{\partial h(m)} = 0, \;\; 0 \le m \le N$$

$\sum_{r=0}^{N} \sum_{n=0}^{M-1} [a_r C(r,\alpha)(n-I)^{r-\alpha} - \sum_{k=0}^{N} a_r h(k)(n-$ $k)r2=0$

$\sum_{k=0}^{N} h(k) [\sum_{r=0}^{N} \sum_{n=0}^{M-1} a_r (n-m)^r (n-k)^r =$ $r=0Nn=0M-1arCr,\alpha n-Ir-\alpha(n-m)r$ (17)

$$\sum_{k=0}^{N} R(m,k) h(k) = T(m,\alpha) \quad 0 \le m \le N \quad (18)$$

where

$$R(m,k) = \sum_{r=0}^{N} \sum_{n=0}^{M-1} a_r (n-m)^r (n-k)^r \quad (19)$$
$$T(m,\alpha) = \sum_{r=0}^{N} \sum_{n=0}^{M-1} a_r C(r,\alpha)(n-I)^{r-\alpha}(n-m)^r \quad (20)$$

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Solving eq. (18) gives the filter coefficient $h(k)$. In the next section, two examples are used to evaluate the performance of least square fractional order differentiator.

IV. EXPERIMENTS AND ANALYSIS

In this section, the results attained for fractional order differentiator based on the least square technique are discussed. To validate the effectiveness of the least square method, two examples are solved. The main purpose of this design method is to decrease the effect of error between the ideal and the desired response of fractional order differentiator. To estimate the performance of the least square fractional order differentiator, the integral square error function of frequency response can be written as

$$\varepsilon_m = \int_0^{\pi} \left| H(e^{j\omega}) - H(e^{-j\omega}) \right|^2 \qquad (21)$$

The error is calculated in the frequency range $[0, \pi]$. The Grunwald-Letnikov definition was used with range of frequency $\omega \,\epsilon\, [0, \pi]$ for the calculation of fractional order derivative.

Experiment 1: In this experiment, the design of fractional order differentiator has been given for m = 3, N = 2, M = 3, delay I = 6 and order α = 0.5, the fractional order derivative of the polynomial signal has been calculated. For α = 0.5, eq. (20) can be written as

$$T(3{,}0.5) = \sum_{r=0}^{2} \sum_{n=0}^{2} a_r C(r, 0.5)(n-6)^{0.5}(n-3)^r \qquad (22)$$

The above coefficients $C(r{,}0.5)$ are given by

$$C(r, 0.5) = \frac{\Gamma(r+1)}{\Gamma(r+0.5)} \qquad (23)$$

Substituting eq. (22) into eq. (21), we get

$$\begin{aligned} T(3{,}0.5) = {} & \frac{a_0\Gamma(1)}{\Gamma(0.5)}[(-6)^{-0.5} + (-5)^{-0.5} + (-4)^{-0.5}] \\ & + \frac{a_1\Gamma(2)}{\Gamma(1.5)}[-3(-6)^{0.5} - 2(-5)^{0.5} - (-4)^{0.5}] \\ & + \frac{a_2\Gamma(3)}{\Gamma(2.5)}[9(-6)^{1.5} - 2(-5)^{1.5} \\ & - (-4)^{1.5}] \end{aligned} \qquad (24)$$

For the given m = 3, N = 2, M = 3, delay I = 6 and order α = 0:5, eq. (19) can be written as

$$R(3, k) = \sum_{r=0}^{2} \sum_{n=0}^{2} a_r (n-3)^r (n-k)^r \qquad (25)$$

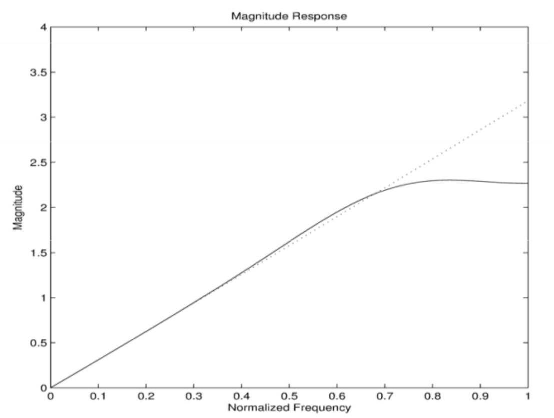

Fig. 1. Magnitude responses of the fractional order FIR differentiators for α = 0:5.

The solid lines are the designed magnitude responses and dotted lines are ideal responses .

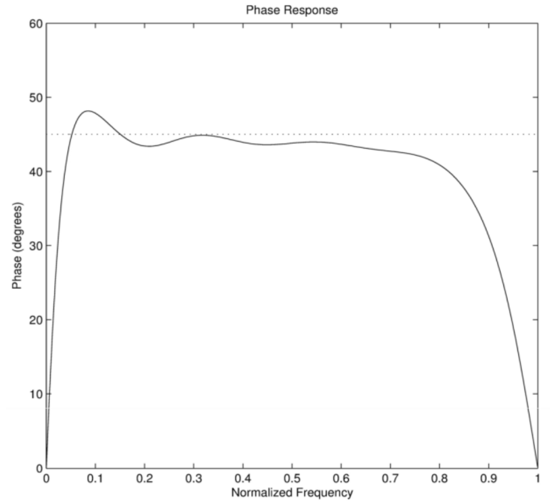

Fig. 2. Phase responses of the fractional order FIR differentiators for α = 0:5.

The solid line is the designed phase responses and dotted lines are ideal responses.

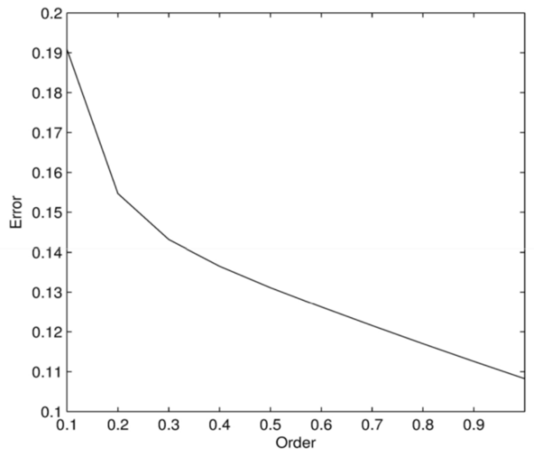

Fig. 3. The integral squared error for the fractional order differentiator H(z) with order α in experiment 1.

Fig. 1 illustrates the magnitude response of the fractional order differentiator of polynomial signal with α = 0:5. The dotted line is the ideal magnitude response ω^{α}. Approximation errors can be decrease by selecting higher value of N and M. Fig. 2 shows the phase response of the designed fractional order differentiators. The dotted line is the ideal phase response 90α. It can be observed that the fractional order α must be selected large enough to minimize the objective error function. Fig. 3 shows the error curve of the projected fractional order differentiator.

Experiment 2: The design Example as given in (Tseng (2001)), where N = 10, M = 11, delay I = 5, and order α = 1; 1:5; 2 is repeated with power function based least square method for the designing of fractional order differentiator. Fig. 4 shows the magnitude response of designed fractional order differentiator and designed example in (Tseng (2001)). The fractional order derivative of the specified polynomial signal can be accurately calculated using proposed method.

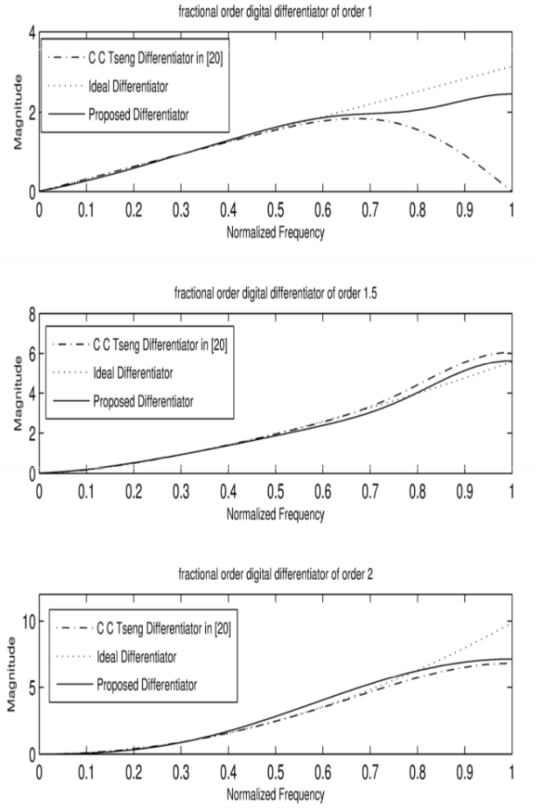

Fig. 4. Magnitude responses of the designed fractional order FIR differentiators. (a). order $\alpha = 1$ (b). order $\alpha = 1.5$ (c). order $\alpha = 2$

V. CONCLUSIONS

In this paper, a fractional order digital differentiator is implemented based on power function and least square technique. The objective error function is measured by the integral squared error function of frequency response, such that the integral squared error can be minimized as much as probable. To validate the effectiveness of the designed method, two design experiments are presented. The results reveal that the least-square technique gives superior performance comparison with present filtering techniques. The results show that the integral squared error is minimized by selecting higher value of truncation length *N*. This technique can also be extended for the design of IIR fractional order differentiator.

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